On the Mathematical Foundation of ABC A Robust Set for Estimating Mechanistic Network Models

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Mathematical Foundation of ABC

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Presentation plan

The four sections and the main references

A mathematical frame for ABC
 A convergence result for ε↓0
 Optimal transport theory in ABC
 Some lower bounds for n→∞



- E. Bernton, P.E. Jacob, M. Gerber, C.P. Robert. Approximate Bayesian computation with the Wasserstein distance. J. R. Statist. Soc. B (2019). Vol. 81, Issue 2, pp. 235–269.
- S.A. Sisson, Y. Fan, M.A. Beaumont. Handbook of Approximate Bayesian Computation. Chapman & Hall/CRC, Handbooks of Modern Statistical Methods, 2019.
- C. Villani. Optimal Transport. Old and New. Springer, 2009.

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Underlying probability space: $(\Omega, \mathcal{A}, \mathbf{P})$. Dimensions: $d_Y, d_Y, n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Observations: $y^{\pm n}(\omega) = y^{\pm n} = (y^{\pm}, \dots, y^n) \in \mathcal{Y}^n, \omega \in \Omega$, where $\mathcal{Y} \subseteq \mathbb{R}^{d_Y}$ has metric a_Y . Predimeters $\mathcal{Y} \in \mathcal{H}$, where $\mathcal{Y} \subseteq \mathbb{R}^{d_Y}$ has metric a_Y .

Given topological spaces X, Y, we denote by: $\mathcal{B}(X)$ the σ -algebra of the Borel subsets of X \mathscr{P} the class of the probability measures on $\mathcal{B}(X)$; $\mathcal{B}(X,Y)$ the class of the measureble functions $(X, \mathcal{B}(X)) \rightarrow (Y, \mathcal{B}(Y))$. We write $\forall d \in \mathcal{H}$ meaning $\forall d \in \mathcal{H} [\pi]$ (for π -a.a. $d \in \mathcal{H}$).

Axiom [A0-a]

The model $\{\mu_{\vartheta}^{n}\}_{\vartheta \in \mathcal{H}}$ is generative meaning that, $\forall \vartheta \in \mathcal{H}$, it's possible to generate how many $z^{1:n} = (z^{1}, \ldots, z^{n}) \in \mathcal{Y}^{n}$ with $z^{1:n} \sim \mu_{\vartheta}^{n}$ we desire.

Pseudo-observations: $z^{1:n} \sim \mu_{\psi}^{n}$ in $\mathcal{Y}^{n}, \widetilde{\forall} \ \psi \in \mathcal{H}$. Deviation measure: \mathcal{D} , pseudo-metric on \mathcal{Y}^{n} .

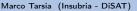
 $\forall \ \vartheta \in \mathcal{H}, \ \mathcal{Y}^n_\vartheta := \{ \ z^{1,n} \in \mathcal{Y}^n \ | \ z^{1,n} \leftarrow \mu^n_\vartheta \}, \ \forall \ \varepsilon > 0, \ D' = \{ \ z^{1,n} \in \mathcal{Y}^n \ | \ \mathcal{D}(y^{1,n}, z^{1,n}) \leq \varepsilon \}$

Axiom [A0-b] (under A0-a)

There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, the two following conditions hold.

 $\int_{\Omega} dG |P_{n}^{2}| \pi(d\theta) > 0 \ (i.e. \neq 0).$

Regarding the whole continuation, we assume that A0 - a and A0 - b wort



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In [] the function $d \rightarrow d_1(\beta_1(\beta_2))$ to be seen as defined $\pi \rightarrow d_2$, indexing to $d\theta(\beta_1(\beta_1,\beta_1))$ [] [] $\mu_1(\beta_2) = (\mu_1(\beta_1) > 0$ () $\mu_2 \geq 0$]

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(i) Choose $\varepsilon \in]0, \varepsilon_0[$. (ii) Draw $\vartheta \in \mathcal{H}$ by π and $z^{1:n} \in \mathcal{Y}^n_{\vartheta}$. (iii) Keep ϑ if, and only if, $z^{1:n} \in D^n_{\varepsilon}$.

ABC posteriors: $\pi_{v^{1:n}}^{\varepsilon} \ll \pi$, $\forall \varepsilon \in]0, \varepsilon_0[$, whose density is proportional to $\mu_{(\cdot)}^n[D_{\varepsilon}^n]$: $\forall B \in \mathcal{B}(H)$,

$$\pi_{y^{\mathbf{1}:n}}^{\varepsilon}[B] = \frac{\int_{B} \mu_{\vartheta}^{n}[D_{\varepsilon}^{n}] \pi(\mathsf{d}\vartheta)}{\int_{\mathcal{H}} \mu_{\vartheta'}^{n}[D_{\varepsilon}^{n}] \pi(\mathsf{d}\vartheta')}$$

Axiom [A0 - c]

For any $Y \in \mathcal{B}(\mathcal{Y}^n)$, $\mu_{(\cdot)}^n[Y] \in \mathscr{B}(\mathcal{H}, [0, 1])$ (coherently w.r.t. A0-b).

Model for the true posterior (under A0-c): for $Y \in \mathcal{B}(\mathcal{Y}^n)$ and $B \in \mathcal{B}(\mathcal{H})$ with $\pi[B] > 0$,

$$\mathsf{P}[\mathsf{Y}|B] \doteq \frac{1}{\pi[B]} \int_{B} \mu_{\vartheta}^{n}[\mathsf{Y}] \, \pi(\mathsf{d}\vartheta).$$

The corresponding posterior: for $Y \in \mathcal{B}(\mathcal{Y}^n)$ and $B \in \mathcal{B}(\mathcal{H})$, whenever it makes sense,

$$\pi[B|Y] = \frac{\int_B \mu_{\vartheta}^n[Y] \pi(\mathsf{d}\vartheta)}{\int_{\mathcal{H}} \mu_{\vartheta'}^n[Y] \pi(\mathsf{d}\vartheta')}.$$

Therefore, the true posterior would be

$$\pi[\cdot|y^{1:n}] \coloneqq \pi[\cdot|\{y^{1:n}\}].$$

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We denote by $\mathbf{m} := \mathbf{m}^{d_{\mathcal{Y}} \cdot n}$ the Lebesgue measure on $\mathcal{B}(\mathbb{R}^{d_{\mathcal{Y}} \cdot n})$.

Axiom [A1]

 $\widetilde{\forall} \ \vartheta \in \mathcal{H}$, the two following conditions hold.

 $\boxed{\quad} \mu_{\theta}^{n} \ll \mathsf{m} \text{ with } f_{\theta}^{n} \coloneqq \mathsf{d} \mu_{\theta}^{n} / \mathsf{d} \mathsf{m} \text{ such that, } \forall z^{1:n} \in \mathcal{Y}^{n} [\mathsf{m}], f_{(+)}^{n}(z^{1:n}) \in \mathscr{B}(\mathcal{H}, \mathbb{R}_{+}).$

 $f_{ij}^{n}(\cdot)$ is continuous and $f_{ij}^{n}(r^{1,n})$ is not π -a.s. identically zero.

1 of A1 implies A0- c while 2 of A1 ensures that $\int_{\mathcal{H}} f_{\vartheta}^{n}(y^{1:n}) \pi(d\vartheta) > 0$ (eventually co).

There exist $\delta, \overline{\varepsilon} \in]0, \infty[$ and $g \in L^1(\pi)$ with $g \ge \delta \ [\pi]$ all such that, $\widetilde{\forall} \ \vartheta \in \mathcal{H}$,

 $\delta \leq \sup_{z^{\mathbf{1}:n}\in D^n_{arepsilon}} f^n_artheta(z^{\mathbf{1}:n}) \leq g(artheta).$

following generalization of A2 and work. Axiom (A2) (under A1) There exist $g \in L^1(\pi)$ with g > 0 [π] and $\varepsilon \in]0, \infty[$ such that, for any $\varepsilon \in]0, \mathbb{Z}_{\varepsilon}$ there exists $\delta_{\varepsilon} \in]0, \infty[$ such that, $\forall \vartheta \in \mathcal{H}$,

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Axiom $[\widetilde{A2}^1]$ (under A1) There exist $g \in L^1(\pi)$ with g > 0 [π] and $\varepsilon \in]0, \infty[$ such that, for any $\varepsilon \in]0, \mathbb{Z}_r$, there exists $\delta_r \in]0, \infty[$ such that, $\forall \ \vartheta \in \mathcal{H}$,

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 $\widetilde{\forall} \ \vartheta \in \mathcal{H}$, the two following conditions hold.

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 $f^n_{\vartheta}(\cdot)$ is continuous and $f^n_{(\cdot)}(y^{1:n})$ is not π -a.s. identically zero.

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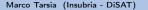
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Axiom [A21 (under A1) There exist $g \in L^1(\pi)$ with g > 0 [π] and $\ell \in]0, \infty$ [such that, for any $\varepsilon \in [0, \infty]$, there exists $\delta_{\varepsilon} \in [0, \infty]$ such that, $\forall \vartheta \in \mathcal{H}$,

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The implies $f^{(n)}(x^{(n)}) \in L^1(\pi)$ with $L^1(\pi)$ -norm lower of equal than $\|v\|_{L^\infty} = \|g\|_{L^\infty}^{\infty}$. As a following generalization of A2 could work. **Axiom [A2]** (under A1) There exist $g \in L^1(\pi)$ with g > 0 [π] and $f \in]0, \infty$ [such that, for any $\varepsilon \in [0, \infty]$, there exists $\delta \in [0, \infty]$ such that, $\forall \vartheta \in \mathcal{H}$,

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A2 would imply 2 of A0-b employing any ε₀ ∈]0, ε̄].
 A2 implies fⁿ₍₋₎(y^{1,n}) ∈ L¹(π) with L¹(π)-norm lower or equal than ||g||₁ = ||g||_{L¹(π)}.
 Even the following generalization of A2 would work.
 Aximple (A2) (under the cost of C (n)) with g = 0 [n] and d ∈ [0, ∞] such that, for any g ∈ [0, ∞] there exists a ∈ [0, ∞] such that y ϑ ∈ H.

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<<p>Image: 1

 $\widetilde{\forall} \ \vartheta \in \mathcal{H}, \ \mathcal{D}(y^{1:n}, \cdot)^{-1}(0) \subseteq f_{\vartheta}^n(\, \cdot\,)^{-1}\big(f_{\vartheta}^n(y^{1:n})\big).$

In particular, if ${\cal D}$ is an actual metric, then A3 trivially holds.

Proposition

Under assumptions A1, A2 and A3, the three following conditions hold.



The ABC posterior strongly converges to the true posterior as $e \downarrow 0$: V $B \in \mathcal{B}(\mathcal{H})$

 $\pi^{\ell_{1m}}_{f^{1m}}[B] \rightarrow \pi[B|_{f^{1m}}] \cup as c \downarrow 0.$

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The ABC rejection algorithm and the ABC posterior are well defined for any $\varepsilon \in]0, \varepsilon_0 \vee \overline{\varepsilon}[$. The true posterior $\pi[\cdot | y^{\log}]$ makes sense and takes the following expression: $\forall \in B(\mathcal{H})$.

$$\pi(P(y^{1,0}) = \frac{\int_{B} G(y^{1,0}) \pi(dx^{0})}{\int_{B} G(y^{1,0}) \pi(dx^{0})}$$

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$$\pi[B|\mathbf{y}^{1:n}] = \frac{\int_B f_{\vartheta}^n(\mathbf{y}^{1:n}) \,\pi(\mathrm{d}\vartheta)}{\int_{\mathcal{H}} f_{\vartheta'}^n(\mathbf{y}^{1:n}) \,\pi(\mathrm{d}\vartheta')}$$

The ABC posterior strongly converges to the true posterior as $\varepsilon \downarrow 0$: $\forall B \in \mathcal{B}(\mathcal{H})$,

$$\pi^{\varepsilon}_{y^{1:n}}[B] o \pi[B|y^{1:n}] \quad \text{as } \varepsilon \downarrow 0.$$

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 $\widetilde{\forall} \ \vartheta \in \mathcal{H}, \ \mathcal{D}(y^{1:n}, \cdot)^{-1}(0) \subseteq f_{\vartheta}^{n}(\cdot)^{-1}(f_{\vartheta}^{n}(y^{1:n})).$

In particular, if \mathcal{D} is an actual metric, then A3 trivially holds.

Proposition

Under assumptions A1, A2 and A3, the three following conditions hold.

- **1** The ABC rejection algorithm and the ABC posterior are well defined for any $\varepsilon \in]0, \varepsilon_0 \lor \overline{\varepsilon}[.$
- **2** The true posterior $\pi[\cdot | y^{1:n}]$ makes sense and takes the following expression: $\forall B \in \mathcal{B}(\mathcal{H})$,

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Marco Tarsia (Insubria - DiSAT)

Let's visualize $(\mathcal{Y}, \varrho_{\mathcal{Y}})$ as a separable and complete metric space, thus also a Radon space, i.e. any element in $\mathscr{P}(\mathcal{Y})$ is a Radon probability measure (outer regular on Borel subsets and inner regular on open subsets); and let's choose an unit cost function $c: \mathcal{Y} \times \mathcal{Y} \to [0, \infty]$ which is lower semicontinuous (so Borel measurable) and a parameter $p \in [1, \infty[$ of summability.

We denote by $\mathscr{P}_{p}(\mathcal{Y})$ the subclass of $\mathscr{P}(\mathcal{Y})$ whose elements have finite p -th moment.

Kantorovich's formulation. For $\mu, \nu \in \mathscr{P}_p(\mathcal{Y})$, consider the subclass $\Gamma(\mu, \nu)$ of $\mathscr{P}(\mathcal{Y} \times \mathcal{Y})$ whose elements γ are the couplings with marginals μ and ν . Then the Kantorovich's formulation of the optimal transport problem related to $(\mathcal{Y}, \varrho_{\mathcal{Y}})$, c and ρ is

 $\mathcal{K}(\mu,\nu) \doteq \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathcal{Y} \times \mathcal{Y}} c(y,y') \, \mathrm{d}\gamma(y,y').$

It can be shown that there exists a minimizer $\gamma^* \in \Gamma(\mu, \nu)$ for such a problem which could be determined by means of gradient descent algorithms.

Example

For $c = (\varrho_{\mathcal{Y}})^p$, \mathcal{K} coincides with the *p*-power of the Wasserstein distance: $\mathcal{K} = \mathcal{W}_p^p$.

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$$\mathcal{M}(\mu,\nu) \doteq \inf_{T \in \mathsf{T}(\mu,\nu)} \int_{\mathcal{V}} c(y,T(y)) \, \mu(\mathrm{d} y).$$

Example

Assume dy = 1 and $\mathcal{Y} = \mathbb{R}$ with ϱy equal to the Euclidean metric. If there exists a function $\varphi \colon \mathbb{R} \to \mathbb{R}$ which is convex and such that $c(y, y') = \varphi(y - y')$, $y, y' \in \mathbb{R}$, then, for $\mu, \nu \in \mathscr{P}_p(\mathbb{R})$ with μ not atomic, the function $T^* := F_{\nu}^{-1} \circ F_{\mu} \in \mathsf{T}(\mu, \nu)$ is an optimal transport map w.r.t. the Monge's formulation (the unique if φ is strictly convex) and the following identity holds:

$$\mathcal{M}(\mu,\nu) \equiv \int_{\mathbb{R}} \varphi(y - T^*(y)) \, \mu(\mathrm{d}y) = \int_0^1 \varphi(F_{\mu}^{-1}(t) - F_{\nu}^{-1}(t)) \, \mathrm{d}t.$$

Radon Symptotic. For any $\mu, \nu \in \mathscr{P}_{p}(\mathcal{Y}), \ \varrho_{\mathcal{R}}(\mu, \nu) \doteq \sup_{h \in C^{0}(\mathcal{Y}, I-1, 1) \mid \mathcal{Y}} h(y) \ (\mu - \nu)(dy) \ defines a$

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Radon symmetric. For any $\mu, \nu \in \mathscr{P}_{p}(\mathcal{Y}), \ \varrho_{\mathcal{R}}(\mu, \nu) \doteq \sup_{h \in C^{\alpha}(Y, I-1, 1)} \int_{\mathcal{Y}} h(y) (\mu - \nu)(dy)$ defines a

metric on $\mathscr{P}_{0}(\mathcal{Y})$ whose notion of convergence corresponds with the total variation of

$$\mathcal{M}(\mu, \nu) \doteq \inf_{\mathcal{T} \in \mathsf{T}(\mu, \nu)} \int_{\mathcal{Y}} c(y, \mathcal{T}(y)) \, \mu(\mathsf{d} y).$$

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Assume $d_{\mathcal{Y}} = 1$ and $\mathcal{Y} = \mathbb{R}$ with $\varrho_{\mathcal{Y}}$ equal to the Euclidean metric. If there exists a function $\varphi \colon \mathbb{R} \to \mathbb{R}$ which is convex and such that $c(y, y') = \varphi(y - y'), y, y' \in \mathbb{R}$, then, for $\mu, \nu \in \mathscr{P}_{p}(\mathbb{R})$ with μ not atomic, the function $\mathcal{T}^* := F_{\nu}^{-1} \circ F_{\mu} \in \mathsf{T}(\mu, \nu)$ is an optimal transport map w.r.t. the Monge's formulation (the unique if φ is strictly convex) and the following identity holds:

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Mathematical Foundation of ABC

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$$\mathcal{M}(\mu,
u) \doteq \inf_{\mathcal{T} \in \mathsf{T}(\mu,
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Marco Tarsia (Insubria - DiSAT) Math

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Deviation measure of distributions: $\forall n \in \mathbb{N}^*$, $\forall y^{1:n} \in \mathcal{Y}^n$, $\forall \vartheta \in \mathcal{H}$, $\forall z^{1:n} \in \mathcal{Y}^n_{\theta}$, we univocally associate an element in $\mathscr{P}(\mathcal{Y})$, possibly in $\mathscr{P}_p(\mathcal{Y})$, $\mu_n \equiv \mu_{y^{1:n}}$ to $y^{1:n}$ and $\mu_{\vartheta,n} \equiv \mu_{\vartheta,z^{1:n}}$ to $z^{1:n}$, and we select a pseudo-distance \mathcal{T} on $\mathscr{P}(\mathcal{Y})$, possibly on $\mathscr{P}_p(\mathcal{Y})$.

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Axiom [B0]

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 $= \mathcal{Y}_{0}^{*} \in \mathcal{B}(\mathcal{Y}^{*}).$

 $\mathbb{Y} \ y^{1,n} \in \mathcal{Y}'$, the function $x^{1,n} \mapsto \mathcal{T}(\mu_n, \mu_{d,n})$ belongs to $\mathcal{B}(\mathcal{Y}'_n, \mathbb{R}_n)$

 $\mathbb{V}[y^{2:n} \in \mathcal{Y}^n \text{ and } \forall e \in]0, e_0[.$

 $|\{z^{1,0} \in \mathcal{N}_{0}^{n} | | | z^{1,0} \in \mathcal{N}_{0}^{n} | | T(\mu_{n}, \mu_{n}, n) \le c \}|$

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 $\boxed{[]} \forall y^{1n} \in \mathcal{Y}_{c}, \text{ the function } z^{1n} \mapsto \mathcal{T}(\mu_{n}, \mu_{d,n}) \text{ belongs to } \mathscr{B}(\mathcal{Y}_{d}^{n}, \mathbb{R}_{+})$

 $\forall y^{\pm in} \in \mathcal{Y}^n$ and $\forall z \in [0, z_0]$.

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■ $\forall y^{1:n} \in \mathcal{Y}^{n}$, the function $z^{1:n} \mapsto \mathcal{T}(\mu_n, \mu_{\partial, n})$ belongs to $\mathscr{B}(\mathcal{Y}^{n}_{\partial}, \mathbb{R}_+)$. ■ $\forall y^{1:n} \in \mathcal{Y}^{n}$ and $\forall z \in [0, z_0]$.

 $|\{z^{1,n} \in \mathcal{Y}_{0}^{n} | | \{z^{1,n} \in \mathcal{Y}_{0}^{n} | | T(\mu_{n}, \mu_{n}, n) \leq \epsilon \}|\}$



 $\forall n \in \mathbb{N}^*$, we write $\widetilde{\forall} y^{1:n} \in \mathcal{Y}^n$ meaning to vary of $y^{1:n}(\omega) \equiv y^{1:n}$ in \mathcal{Y}^n for P-a.a. $\omega \in \Omega$.

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Example

 $\mu_n = \widehat{\mu}_n := n^{-1} \sum_{k=1}^n \delta_{y^k} \text{ and } \mu_{\vartheta,n} = \widehat{\mu}_{\vartheta,n} := n^{-1} \sum_{k=1}^n \delta_{z^k} \text{ (empirical distributions)}.$

Axiom [B0]

∀ n ∈ N* and ∀ ϑ ∈ H, the three following conditions hold.
1 𝔅ⁿ_ϑ ∈ 𝔅(𝔅ⁿ).
2 ∀ 𝗴^{1,n} ∈ 𝔅ⁿ, the function ż^{1,n} ↦ 𝒯(μ_n, μ_{ϑ,n}) belongs to 𝔅(𝔅ⁿ_ϑ, ℝ₊
3 ∀ 𝗴^{1,n} ∈ 𝔅ⁿ and ∀ ε ∈]0, ε₀[,

 $\mu_{\vartheta}^{n}[D_{\varepsilon}^{n}] \geq \mu_{\vartheta}^{n}\left[\left\{ z^{1:n} \in \mathcal{Y}_{\vartheta}^{n} \mid \mathcal{T}(\mu_{n}, \mu_{\vartheta, n}) \leq \varepsilon \right\}\right].$

3 of **b**0 holds if, $\forall n \in \mathbb{N}^*$, $\forall y^{1:n} \in \mathcal{Y}^n$, $\forall \vartheta \in \mathcal{H}$ and $\forall z^{1:n} \in \mathcal{Y}^n_{\theta}$

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∀ y^{1:n} ∈ Yⁿ and ∀ ε ∈]0, ε₀[,

 $\mu_{\vartheta}^{n}[D_{\varepsilon}^{n}] \geq \mu_{\vartheta}^{n} \left[\left\{ z^{1:n} \in \mathcal{Y}_{\vartheta}^{n} \mid \mathcal{T}(\mu_{n}, \mu_{\vartheta, n}) \leq \varepsilon \right\} \right].$

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$$3 \quad \widetilde{\forall} y^{1:n} \in \mathcal{Y}^n \text{ and } \forall \varepsilon \in]0, \varepsilon_0[,$$

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$$\mathbf{\mathcal{Y}}_{\vartheta}^n \in \mathcal{B}(\mathcal{Y}^n).$$

$$\mathbf{\widetilde{\forall}} \ y^{1:n} \in \mathcal{Y}^n, \text{ the function } z^{1:n} \mapsto \mathcal{T}(\mu_n, \mu_{\vartheta, n}) \text{ belongs to } \mathscr{B}(\mathcal{Y}_{\vartheta}^n, \mathbb{R}_+).$$

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 $\mathcal{D}(y^{1:n}, z^{1:n}) \leq \mathcal{T}(\mu_n, \mu_{\vartheta, n}).$

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 $\mathcal{T}(\mu_n,\mu_\star) o \mathsf{0}, \ \mathsf{P} ext{-a.s., as } n o \infty.$

Axiom [B2] (under B1)

 $\forall \ \vartheta \in \mathcal{H}$, there exists unique $\mu_{\vartheta} \in \mathscr{P}(\mathcal{Y})$, possibly in $\mathscr{P}_{p}(\mathcal{Y})$, such that the following occurs.

 $[] \forall n \in \mathbb{N} \text{ and } \forall \theta \in \mathcal{H}, \text{ the function } z^{1n} \mapsto \mathcal{T}(\mu_{\theta,n},\mu_{\theta}) \text{ belongs to } \mathcal{B}(\mathcal{Y}_{0}^{n},\mathbb{R}_{0}).$

There exists $r \in [0, 1]$ such that, $\tilde{\forall} \not a \in \mathcal{H}$ and $\forall e > 0$,

 $\limsup_{v \in \mathcal{V}_{v}} \mu_{\theta}^{s} \big[\big\{ x^{1,v} \in \mathcal{Y}_{\theta}^{s} \mid \mathcal{T} \big\{ \mu_{\theta,v}, \mu_{\theta} \big\} > \varepsilon \big\} \big] \leq \tau.$

There exist $\sigma \in [0, \tau]$ and $c_1 > 0$ such that, $\forall \ \sigma \in \mathcal{H}$ and $\forall \ e \in [0, \epsilon_1]$.

 $m\inf_{\sigma}\mu_{\sigma}^{*}\left[\left\{\left.z^{1,\sigma}\in\mathcal{Y}_{\sigma}^{*}\right|\mathcal{T}\left(\mu_{\sigma,\sigma},\mu_{\sigma}\right)>\varepsilon\right\}\right]\geq\sigma.$

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Axiom [B2] (under B1)

 $\forall \ \vartheta \in \mathcal{H}$, there exists unique $\mu_{\vartheta} \in \mathscr{P}(\mathcal{Y})$, possibly in $\mathscr{P}_{\rho}(\mathcal{Y})$, such that the following occurs.

 $\mathbb{W} \to \mathbb{W}^*$ and $\mathbb{V} \in \mathcal{H}$, the function $z^{kn} \mapsto \mathcal{T}(\mu_{d,n}, \mu_d)$ belongs to $\mathscr{B}(\mathcal{Y}_{0}, \mathbb{R}_{+})$.

There exists $x \in [0, 1]$ such that, $\tilde{\forall} \not a \in \mathcal{H}$ and $\forall x > 0$.

 $\limsup_{n}\mu_0^s\left[\left\{|z^{1,n}\in\mathcal{Y}_0^s\mid \mathcal{T}(\mu_{2,n},\mu_2)>\varepsilon\right\}\right]\leq \tau.$

There exist $\sigma \in [0, \tau]$ and $c_1 > 0$ such that, $\forall d \in \mathcal{X}$ and $\forall e \in [0, c_1]$.

 $\min_{i \in \mathcal{M}} \mu_{\theta}^{s}[\{x^{1,s} \in \mathcal{Y}_{\theta}^{s} \mid \mathcal{T}(\mu_{\theta_{i},n}, \mu_{\theta}) \geq \epsilon\}] \geq \alpha$.

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 \mathbb{Z} V $n \in \mathbb{N}^*$ and \mathbb{Y} $\theta \in \mathcal{H}_{\epsilon}$ the function $z^{1n} \mapsto \mathcal{T}(\mu_{\theta,n},\mu_{\theta})$ belongs to $\mathscr{G}(\mathbb{Y}_{\theta}^*,\mathbb{R}_{\tau})$.

There exists $r \in [0, 1]$ such that, $\forall d \in \mathcal{H}$ and $\forall e > 0$,

 $\lim\sup_{p \in \mathcal{J}_{p}^{k}} \mu_{p}^{k} \left[\left\{ \left| z^{1,p} \in \mathcal{J}_{p}^{k} \right| \mid \mathcal{T}(\mu_{\sigma}, \mu_{\sigma}) > \alpha \right\} \right] \leq \tau_{\tau} \cdot \left[\left\{ \left| z^{1,p} \in \mathcal{J}_{p}^{k} \right| \mid \mathcal{T}(\mu_{\sigma}, \mu_{\sigma}) > \alpha \right\} \right] \leq \tau_{\tau} \cdot \left[\left\{ \left| z^{1,p} \in \mathcal{J}_{p}^{k} \right| \mid z^{1,p} \in \mathcal{J}_{p}^{k} \right\} \right] \leq \tau_{\tau} \cdot \left[\left| z^{1,p} \in \mathcal{J}_{p}^{k} \right| \mid z^{1,p} \in \mathcal{J}_{p}^{k} \right]$

There exist $v \in [0, r]$ and $e_i > 0$ such that, $\forall v \in \mathcal{H}$ and $\forall e \in [0, e_i]$,

 $\min_{a} \mu_{\theta}^{s}[\{x^{1,a} \in \mathcal{Y}_{\theta}^{s} \mid \mathcal{T}(\mu_{\theta_{1}a_{1}}, \mu_{\theta}) > \varepsilon\}] \geq \sigma.$

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 $\mathcal{T} \in \mathbb{N}^*$ and $\forall \ \theta \in \mathcal{H}$, the function $z^{\perp n} \mapsto \mathcal{T}(\mu_{\theta,n}, \mu_{\theta})$ belongs to $\mathscr{B}(\mathcal{Y}^n_{\theta}, \mathbb{R}_+)$

 $\operatorname{msup}_{\sigma} \mu_{\theta}^{*}[\{z^{1,\sigma} \in \mathcal{Y}_{0}^{\sigma} | T(\mu_{\sigma}, \mu_{\theta}) \geq \varepsilon\}] \leq \tau \tau$

There exist $\sigma \in [0, \tau]$ and $e_1 > 0$ such that, $\forall \phi \in \mathcal{H}$ and $\forall e \in [0, e_1]$

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4 There exist $\sigma \in [0, au]$ and $\varepsilon_1 > 0$ such that, $\forall \ \vartheta \in \mathcal{H}$ and $\forall \ \varepsilon \in]0, \varepsilon_1[$,

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- $2 \quad \forall \ n \in \mathbb{N}^* \text{ and } \widetilde{\forall} \ \vartheta \in \mathcal{H}, \text{ the function } z^{1:n} \mapsto \mathcal{T}(\mu_{\vartheta,n}, \mu_{\vartheta}) \text{ belongs to } \mathscr{B}(\mathcal{Y}^n_{\vartheta}, \mathbb{R}_+).$
- 3 There exists $\tau \in [0, 1[$ such that, $\widetilde{\forall} \ \vartheta \in \mathcal{H}$ and $\forall \ \varepsilon > 0$,

 $\limsup_{n} \mu_{\vartheta}^{n} \left[\left\{ z^{1:n} \in \mathcal{Y}_{\vartheta}^{n} \mid \mathcal{T}(\mu_{\vartheta,n},\mu_{\vartheta}) > \varepsilon \right\} \right] \leq \tau.$

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- **1** For any $n \in \mathbb{N}^*$, $\omega \mapsto \mathcal{T}(\mu_n, \mu_*)$ is \mathcal{A} -measurable as a function from Ω to \mathbb{R}_+ .
- 2 $\mathcal{T}(\mu_n, \mu_\star) \rightarrow 0$, **P**-a.s., as $n \rightarrow \infty$.

Axiom [B2] (under B1)

 $\widetilde{\forall} \ \vartheta \in \mathcal{H}$, there exists unique $\mu_{\vartheta} \in \mathscr{P}(\mathcal{Y})$, possibly in $\mathscr{P}_{p}(\mathcal{Y})$, such that the following occurs.

- **1** The function $\vartheta \mapsto \mathcal{T}(\mu_{\vartheta}, \mu_{\star})$ belongs to $\mathscr{B}(\mathcal{H}, \mathbb{R}_+)$.
- $2 \quad \forall \ n \in \mathbb{N}^* \text{ and } \widetilde{\forall} \ \vartheta \in \mathcal{H}, \text{ the function } z^{1:n} \mapsto \mathcal{T}(\mu_{\vartheta,n}, \mu_{\vartheta}) \text{ belongs to } \mathscr{B}(\mathcal{Y}^n_{\vartheta}, \mathbb{R}_+).$
- **3** There exists $\tau \in [0, 1[$ such that, $\forall \vartheta \in \mathcal{H}$ and $\forall \varepsilon > 0$,

 $\limsup_{n} \mu_{\vartheta}^{n} \left[\left\{ z^{1:n} \in \mathcal{Y}_{\vartheta}^{n} \mid \mathcal{T}(\mu_{\vartheta,n},\mu_{\vartheta}) > \varepsilon \right\} \right] \leq \tau.$

4 There exist $\sigma \in [0, \tau]$ and $\varepsilon_1 > 0$ such that, $\widecheck{\forall} \ \vartheta \in \mathcal{H}$ and $\forall \ \varepsilon \in]0, \varepsilon_1[$,

$$\liminf_{n} \mu_{\vartheta}^{n} \left[\left\{ z^{1:n} \in \mathcal{Y}_{\vartheta}^{n} \mid \mathcal{T}(\mu_{\vartheta,n}, \mu_{\vartheta}) > \varepsilon \right\} \right] \geq \sigma.$$

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3 of B2 is equivalent to any version of that in which an upper bound for ε is imposed.

Furthermore if, $\forall \vartheta \in \mathcal{H}$ and $\forall \varepsilon > 0$, $\mu_{\vartheta}^{n}[\mathcal{T}(\mu_{\vartheta,n}, \mu_{\vartheta}) > \varepsilon] \to 0$ as $n \to \infty$ (shortly put), then any $\tau \in [0, 1]$ satisfies 3 of B2 while only $\sigma = 0$ but any $\varepsilon_{1} > 0$ fulfill 4 of B2.

Axiom [B3] (under 1 and 2 of B2)

There exists $\vartheta_{\star} \in \mathcal{H}$ which minimizes $\vartheta \mapsto \mathcal{T}(\mu_{\vartheta}, \mu_{\star})$ over \mathcal{H} : simbolically,

 $\vartheta_{\star} \in \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}(\mu_{(\,\cdot\,)},\mu_{\star}).$

We denote $\varepsilon_* \doteq \mathcal{T}(\mu_{\vartheta_*}, \mu_*) = \min_{\mathcal{H}} \mathcal{T}(\mu_{(*)}, \mu_*) \geq 0$ and, $\widetilde{\forall} \ \vartheta \in \mathcal{H}, \ T_{\vartheta} \coloneqq \mathcal{T}(\mu_{\vartheta}, \mu_*) \geq \varepsilon_{*,*}$

Axiom [B4] (under B3)

There exist a neighborhood $U_* \subset \mathcal{H}$ of ϑ_* , a connected neighborhood $I_0 \subset \mathbb{R}_+$ of zero and a strictly increasing function $\psi : I_0 \to \mathbb{R}_+$ all such that, $\widetilde{\forall} \ \vartheta \in U_*$,

$$\mathcal{T}_{artheta} - arepsilon_{\star} \leq \psi ig(arrho_{\mathcal{H}}(artheta, artheta_{\star}) ig).$$

We write "for any $(y^{1,n})_n$ " meaning to vary of $(y^{1,n}(\omega))_n \equiv (y^{1,n})_n$, with $y^{1,n}(\omega) \equiv (y^{1,n})_n$ for any $e \in \mathbb{N}^*$, w.r.t. a $\omega \in \Omega$. Lastly, for $\varepsilon > 0$, we denote by ε^- any element of $[0, \varepsilon)$.

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We denote $\varepsilon_\star \doteq \mathcal{T}(\mu_{\vartheta_\star}, \mu_\star) = \min_{\mathcal{H}} \mathcal{T}(\mu_{(+)}, \mu_\star) \geq 0$ and, $\forall \ \vartheta \in \mathcal{H}, \ \mathcal{T}_\vartheta \coloneqq \mathcal{T}(\mu_{\vartheta}, \mu_\star) \geq \varepsilon_\star$

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Under assumptions B0, B1, 1, 2 and 3 of B2 and B3, the following occurs so far as $\varepsilon_* < \varepsilon_0$, for $\varepsilon \in]0, \varepsilon_0 - \varepsilon_*[, (y^{1:n})_n \text{ with } n \equiv n_{\varepsilon} \text{ large enough and with probability } P \text{ going to } 1 \text{ as } n \to \infty.$

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Under assumptions B0, B1, 1, 2 and 3 of B2 and B3, the following occurs so far as $\varepsilon_{\star} < \varepsilon_{0}$, for $\varepsilon \in]0, \varepsilon_{0} - \varepsilon_{\star}[, (y^{1:n})_{n}$ with $n \equiv n_{\varepsilon}$ large enough and with probability P going to 1 as $n \to \infty$. 1 $\pi_{y^{1:n}}^{\varepsilon_{\star}+\varepsilon} [\mathcal{T}_{(.)} \ge \varepsilon_{\star} + \varepsilon^{-}/3] \ge (1-\tau)\pi [\varepsilon_{\star} + \varepsilon^{-}/3 \le \mathcal{T}_{(.)} \le \varepsilon_{\star} + \varepsilon/3].$

- 2 $\pi_{v^{1:n}}^{\varepsilon_{\star}+\varepsilon} [\mathcal{H} \setminus \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}_{(\cdot)}] \ge (1-\tau) \pi [\varepsilon_{\star} < \mathcal{T}_{(\cdot)} \le \varepsilon_{\star} + \varepsilon/3].$
- 3 Under assumption 4 of B2, let's suppose that in 3 of B0 the equality holds and that $\varepsilon_* < \varepsilon_1/2$.

 $T_{(1)} \le c_{1} + 5\epsilon/3 + \epsilon_{7} |T_{(1)} \ge c_{2} + 5\epsilon/3 > 0$

42176.2 c + c / a | 2 ⁻¹ - c | a + c / a S / c | S c + c / a |

Under assumption B4, for any $\zeta \in b \setminus \{0\}$ and r > 0 small enough,

 $(2^{+1})^{\circ}[e_{H}(\cdot, \delta_{*}) \ge t] \ge \pi (2^{+1})^{\circ}[T_{(*)} \ge c_{*} + \psi(0)]$

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Under assumptions B0, B1, 1, 2 and 3 of B2 and B3, the following occurs so far as $\varepsilon_{\star} < \varepsilon_{0}$, for $\varepsilon \in]0, \varepsilon_{0} - \varepsilon_{\star}[, (y^{1:n})_{n}$ with $n \equiv n_{\varepsilon}$ large enough and with probability P going to 1 as $n \to \infty$. 1 $\pi_{y_{1:n}}^{\varepsilon_{\star}+\varepsilon} [\mathcal{T}_{(.)} \ge \varepsilon_{\star} + \varepsilon^{-}/3] \ge (1 - \tau) \pi [\varepsilon_{\star} + \varepsilon^{-}/3 \le \mathcal{T}_{(.)} \le \varepsilon_{\star} + \varepsilon/3].$ 2 $\pi_{y_{1:n}}^{\varepsilon_{\star}+\varepsilon} [\mathcal{H} \setminus \arg \min_{\mathcal{H}} \mathcal{T}_{(.)}] \ge (1 - \tau) \pi [\varepsilon_{\star} < \mathcal{T}_{(.)} \le \varepsilon_{\star} + \varepsilon/3].$

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 $= (1 - \sigma)\pi \left[\mathcal{T}_{i,1} \leq \epsilon_i + 5\epsilon/3 \right] + \tau\pi \left[\mathcal{T}_{i,1} > \epsilon_i + 5\epsilon/3 \right] > 0$

Under assumption B4, for any $\zeta \in \mathfrak{h} \setminus \{0\}$ and r > 0 small enough,

 $2^{n+1}[e_n(-,v_i) \ge r] \ge \pi 2^{n+1}[T_{i,j} \ge c_i + v(i)].$

Under assumptions B0, B1, 1, 2 and 3 of B2 and B3, the following occurs so far as $\varepsilon_* < \varepsilon_0$, for $\varepsilon \in]0, \varepsilon_0 - \varepsilon_*[, (y^{1:n})_n \text{ with } n \equiv n_{\varepsilon} \text{ large enough and with probability } \mathbf{P} \text{ going to } 1 \text{ as } n \to \infty.$

$$1 \quad \pi_{v^{1:n}}^{\varepsilon_{\star}+\varepsilon} \left[\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_{\star} + \varepsilon^{-}/3 \right] \geq (1-\tau) \, \pi \left[\varepsilon_{\star} + \varepsilon^{-}/3 \leq \mathcal{T}_{(\,\cdot\,)} \leq \varepsilon_{\star} + \varepsilon/3 \right].$$

$$2 \quad \pi_{\mathbf{y}^{\mathbf{1}:n}}^{\varepsilon_{\star}+\varepsilon} \left[\mathcal{H} \setminus \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}_{(\cdot)} \right] \geq (1-\tau) \, \pi \left[\varepsilon_{\star} < \mathcal{T}_{(\cdot)} \leq \varepsilon_{\star} + \varepsilon/3 \right].$$

3 Under assumption 4 of B2, let's suppose that in 3 of B0 the equality holds and that $\varepsilon_* < \varepsilon_1/2$. Then, for any $\varepsilon \in]0, \varepsilon_0 - \varepsilon_*[$ even more enough small,

$$\lambda_{\varepsilon} \coloneqq (1 - \sigma) \pi \left[\mathcal{T}_{(.)} \le \varepsilon_{\star} + 5\varepsilon/3 \right] + \tau \pi \left[\mathcal{T}_{(.)} > \varepsilon_{\star} + 5\varepsilon/3 \right] > 0$$

and

$$\pi_{j^{1:n}}^{\varepsilon_*+\varepsilon} \left[\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_* + \varepsilon^-/3 \right] \geq \frac{1-\tau}{\lambda_{\varepsilon}} \pi \left[\varepsilon_* + \varepsilon^-/3 \leq \mathcal{T}_{(\,\cdot\,)} \leq \varepsilon_* + \varepsilon/3 \right].$$

Under assumption B4, for any $\zeta \in \mathfrak{h} \setminus \{0\}$ and r > 0 small enough,

 $(2^{+*}_{2^{+}}[e_{\ell}(\cdot, \vartheta_{\ell}) \ge r] \ge r(2^{+*}_{2^{+}}[\mathcal{T}_{(\cdot)} \ge c_{\ell} + \vartheta(0)]$

Under assumptions B0, B1, 1, 2 and 3 of B2 and B3, the following occurs so far as $\varepsilon_* < \varepsilon_0$, for $\varepsilon \in]0, \varepsilon_0 - \varepsilon_*[, (y^{1:n})_n \text{ with } n \equiv n_{\varepsilon} \text{ large enough and with probability } \mathbf{P} \text{ going to } 1 \text{ as } n \to \infty.$

$$1 \quad \pi_{v^{1:n}}^{\varepsilon_{\star}+\varepsilon} \left[\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_{\star} + \varepsilon^{-}/3 \right] \geq (1-\tau) \, \pi \left[\varepsilon_{\star} + \varepsilon^{-}/3 \leq \mathcal{T}_{(\,\cdot\,)} \leq \varepsilon_{\star} + \varepsilon/3 \right].$$

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Under assumption B4, for any $\zeta \in \mathfrak{h} \setminus \{0\}$ and r > 0 small enough,

 $(e_{2n})^{*} [e_{2n}(\cdots, v_n) \ge r] \ge \pi (e_{2n})^{*} [\mathcal{T}_{(1)} \ge c_n + \psi(0)].$

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$$\pi_{j^{1:n}}^{\varepsilon_{\star}+\varepsilon}\left[\mathcal{T}_{(\,\cdot\,)}\geq\varepsilon_{\star}+\varepsilon^{-}/3\right]\geq\frac{1-\tau}{\lambda_{\varepsilon}}\,\pi\left[\varepsilon_{\star}+\varepsilon^{-}/3\leq\mathcal{T}_{(\,\cdot\,)}\leq\varepsilon_{\star}+\varepsilon/3\right].$$

Under assumption B4, for any $\zeta \in I_0 \setminus \{0\}$ and r > 0 small enough,

 $\pi_{\boldsymbol{\gamma}^{\mathtt{lin}}}^{\varepsilon_{\star}+\varepsilon} \big[\varrho_{\mathcal{H}}(\,\cdot\,,\vartheta_{\star}) \geq r \big] \geq \pi_{\boldsymbol{\gamma}^{\mathtt{lin}}}^{\varepsilon_{\star}+\varepsilon} \big[\,\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_{\star} + \psi(\zeta) \big]$

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and

$$\pi_{y^{\mathbf{1}:n}}^{\varepsilon_{\star}+\varepsilon}\left[\mathcal{T}_{(\,\cdot\,)}\geq\varepsilon_{\star}+\varepsilon^{-}/3\right]\geq\frac{1-\tau}{\lambda_{\varepsilon}}\,\pi\left[\varepsilon_{\star}+\varepsilon^{-}/3\leq\mathcal{T}_{(\,\cdot\,)}\leq\varepsilon_{\star}+\varepsilon/3\right].$$

Under assumption B4, for any $\zeta \in I_0 \setminus \{0\}$ and r > 0 small enough,

 $\pi_{arphi^{1,n}}^{arepsilon_{\star}+arepsilon}\left[arepsilon_{\mathcal{H}}(\,\cdot\,,artheta_{\star})\geq r
ight]\geq\pi_{arphi^{1,n}}^{arepsilon_{\star}+arepsilon}\left[\mathcal{T}_{(\,\cdot\,)}\geqarepsilon_{\star}+\psi(\zeta)
ight]$

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$$\pi_{y^{1:n}}^{\varepsilon_{\star}+\varepsilon}\left[\mathcal{T}_{(\,\cdot\,)}\geq\varepsilon_{\star}+\varepsilon^{-}/3\right]\geq\frac{1-\tau}{\lambda_{\varepsilon}}\,\pi\left[\varepsilon_{\star}+\varepsilon^{-}/3\leq\mathcal{T}_{(\,\cdot\,)}\leq\varepsilon_{\star}+\varepsilon/3\right].$$

4 Under assumption B4, for any $\zeta \in I_0 \setminus \{0\}$ and r > 0 small enough,

$$\pi_{y^{1:n}}^{\varepsilon_{\star}+\varepsilon} \left[\varrho_{\mathcal{H}}(\,\cdot\,,\vartheta_{\star}) \geq r \right] \geq \pi_{y^{1:n}}^{\varepsilon_{\star}+\varepsilon} \left[\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_{\star} + \psi(\zeta) \right]$$

for which lower bounds of a and eventually c hold if also ζ is small enough.

Marco Tarsia (Insubria - DiSAT)

Under assumptions B0, B1, 1, 2 and 3 of B2 and B3, the following occurs so far as $\varepsilon_* < \varepsilon_0$, for $\varepsilon \in]0, \varepsilon_0 - \varepsilon_*[, (y^{1:n})_n \text{ with } n \equiv n_{\varepsilon} \text{ large enough and with probability } \mathbf{P} \text{ going to } 1 \text{ as } n \to \infty.$

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3 Under assumption 4 of B2, let's suppose that in 3 of B0 the equality holds and that $\varepsilon_* < \varepsilon_1/2$. Then, for any $\varepsilon \in]0, \varepsilon_0 - \varepsilon_*[$ even more enough small,

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Axiom [A2'] (under A1)

There exist $\delta, \varepsilon' \in]0, \infty[$ and $g \in L^1(\pi)$ with $g \ge \delta$ $[\pi]$ all such that, $\forall \ \vartheta \in \mathcal{H}$ and $\forall (z^{1:n})_n$ with $z^{1:n} \in D_{\varepsilon'}^n$ for any $n \in \mathbb{N}^*$,

 $\delta \leq \liminf_n f_{\vartheta}^n(z^{1:n}) \quad \text{and} \quad \limsup_n f_{\vartheta}^n(z^{1:n}) \leq g(\vartheta).$

Proposition

Under assumptions B0, B1, 1 and 2 of B2, B3, A1 and A2', the following occurs so far as $\varepsilon_* < \varepsilon_0 \wedge \varepsilon'$ and for $\varepsilon \in]0, \varepsilon_0 \wedge \varepsilon' - \varepsilon_*[$ and P-a.a. $(y^{1:n})_n$.

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Proposition

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For any
$$\zeta > 0$$
, $\pi_{g^{\pm}n}^{e_{\pm}+e}[\mathcal{T}(\cdot) \ge e_{+} + \zeta] \ge \frac{1}{\|\|g\|\|_{1}}\pi[\mathcal{T}(\cdot) \ge e_{+} + \zeta]$.
 $\|\|\pi_{g}^{e_{\pm}e_{\pm}}\|_{\mathcal{H}} \wedge \operatorname{arg} \min_{\mathcal{H}}\mathcal{T}(\cdot)\| \ge \frac{\delta}{\|\|g\|\|_{1}}\pi[\mathcal{H} \wedge \operatorname{arg} \min_{\mathcal{H}}\mathcal{T}(\cdot)]$.



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Under assumptions B0, B1, 1 and 2 of B2, B3, A1 and A2', the following occurs so far as $\varepsilon_{\star} < \varepsilon_0 \wedge \varepsilon'$ and for $\varepsilon \in]0, \varepsilon_0 \wedge \varepsilon' - \varepsilon_{\star}[$ and P-a.a. $(y^{1:n})_n$.

 $= \pi_{y^{1,n}}^{\varepsilon_*+\varepsilon} \left[\mathcal{H} \setminus \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}_{(\cdot,)} \right] \geq \frac{\sigma}{\|\sigma\|_{\cdot}} \pi \left[\mathcal{H} \setminus \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}_{(\cdot,)} \right].$

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Under assumptions B0, B1, 1 and 2 of B2, B3, A1 and A2', the following occurs so far as $\varepsilon_{\star} < \varepsilon_0 \land \varepsilon'$ and for $\varepsilon \in]0, \varepsilon_0 \land \varepsilon' - \varepsilon_{\star}[$ and P-a.a. $(y^{1:n})_n$.

1 For any
$$\zeta > 0$$
, $\pi_{y^{1:n}}^{\varepsilon_{\star}+\varepsilon} [\mathcal{T}_{(\cdot)} \ge \varepsilon_{\star} + \zeta] \ge \frac{\delta}{\|g\|_{1}} \pi [\mathcal{T}_{(\cdot)} \ge \varepsilon_{\star} + \zeta].$

 $2 \quad \pi_{y^{1:n}}^{\varepsilon_{\star}+\varepsilon} \left[\mathcal{H} \setminus \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}_{(\cdot)} \right] \geq \frac{o}{\|g\|_{1}} \pi \left[\mathcal{H} \setminus \operatorname{arg\,min}_{\mathcal{H}} \mathcal{T}_{(\cdot)} \right].$

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$$\begin{array}{l} 1 \quad \text{For any } \zeta > 0, \ \pi_{y^{1:n}}^{\varepsilon_{\star} + \varepsilon} \left[\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_{\star} + \zeta \right] \geq \frac{\delta}{\|g\|_{1}} \ \pi\left[\mathcal{T}_{(\,\cdot\,)} \geq \varepsilon_{\star} + \zeta \right]. \\ 2 \quad \pi_{y^{1:n}}^{\varepsilon_{\star} + \varepsilon} \left[\mathcal{H} \setminus \arg \min_{\mathcal{H}} \mathcal{T}_{(\,\cdot\,)} \right] \geq \frac{\delta}{\|g\|_{1}} \ \pi\left[\mathcal{H} \setminus \arg \min_{\mathcal{H}} \mathcal{T}_{(\,\cdot\,)} \right]. \end{array}$$

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Thanks for your attention!